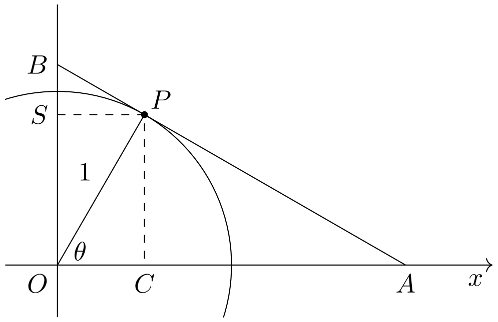
1601. The modulus function is defined over $\mathbb{R}$ as

$$|x| = \begin{cases} -x, & x \in (-\infty, 0) \\ x, & x \in [0, \infty) \end{cases}$$

Show that the graphs $y = |x|$ and $y = 2 - |4 - 2x|$ intersect at precisely one point.

1602. Show that $\lim\limits_{x\to 1} \dfrac{x^2 - 1}{x - 1} = 2$.

1603. Part of the unit circle is shown below. Point P is placed at $(\cos\theta, \sin\theta)$, for $\theta \in (0, 90°)$. A tangent is drawn to the circle at point P, which crosses the x and y axes at A and B. Perpendiculars from P meet the axes at S and C.



Each of the trigonometric ratios $\sin\theta$, $\cos\theta$, $\tan\theta$, $\operatorname{cosec}\theta$, $\sec\theta$ and $\cot\theta$ appears as a length on this diagram. Identify them.

1604. A rigid object is in equilibrium under the action of the following forces in an (x, y) plane:

$2\mathbf{i}$ N at $(0, 1)$,
$2\mathbf{j}$ N at $(-1, 0)$,
$a\mathbf{i} + b\mathbf{j}$ N at some point P.

(a) Explain why the line of action of the third force must pass through $(-1, 1)$.
(b) Find a and b.
(c) Sketch the possible locations of P.

1605. Prove that no quadratic function can have three distinct fixed points.

1606. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$x^2 < 4, \qquad y^2 < 4.$$

1607. Find the following limits, defined in radians.

(a) $\lim\limits_{x\to 0} \dfrac{\sin^3 x}{x^2}$,

(b) $\lim\limits_{x\to 0} \dfrac{\sin^3 x}{x^3}$,

(c) $\lim\limits_{x\to 0} \dfrac{\sin^3 x}{x^4}$.

1608. A particular retirement home has 200 residents. Four in five of the residents are vaccinated against flu. In any given year, 2% of those who have been vaccinated, and 15% of those who haven't been, come down with flu.

(a) Represent this information on a tree diagram.
(b) In any given year, find the expected number who come down with flu among
  i. those not vaccinated,
  ii. everyone.
(c) Show that, among those who come down with flu, the ratio of vaccinated to not vaccinated people is $8 : 15$.

1609. Find, in simplified terms of x, the mean of

$$\left(\sqrt{x} + \sqrt{x-1}\right)^2 \text{ and } \left(\sqrt{x} - \sqrt{x-1}\right)^2.$$

1610. A functional instruction is defined as

$$\mathrm{f} : x \mapsto \frac{\sqrt{x}}{1 + \sqrt{x}}.$$

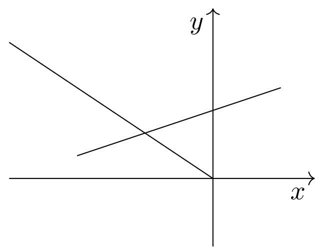
(a) Write down the largest real domain over which f may be defined.
(b) Show that, where f^{-1} is defined, its rule is

$$\mathrm{f}^{-1} : x \longmapsto 1 + \frac{2x - 1}{(x-1)^2}.$$

(c) Show that $y = 1$ is not in the range of f.

1611. Solve $\displaystyle\sum_{j=5}^{7} x^j = 0$.

1612. On the graph below, two line segments defined parametrically by $\mathbf{r}_1 = (3s - 3)\mathbf{i} + (2 - 2s)\mathbf{j}$, for $s \in [0, 1]$ and $\mathbf{r}_2 = t\mathbf{i} + (1 + \frac{1}{3}t)\mathbf{j}$, for $t \in [-2, 1]$ are depicted, to scale.



Verify that the line segments trisect each other.

1613. The random variable X is distributed according to $B(n, p)$. The best normal approximation to X is $N(30, 22.5)$. Find n and p.

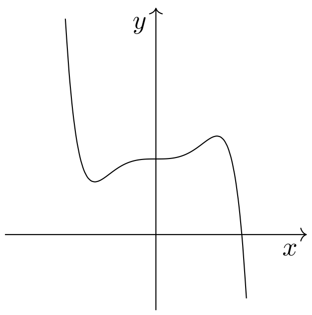
1614. A student writes: "If a rope is extensible, then the tensions exerted by its two ends should not be modelled as equal." State, with a reason, whether this is correct or not.

1615. An isosceles triangle has two sides of length 5, and area 12. Show that there are two possible values for the third length, both of which are integers.

1616. Express 3^{2x+4} in terms of 9^x.

1617. Write down the value of $\displaystyle\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$.

1618. The graph $y = \mathrm{f}(x)$ shown in the diagram is of the function $\mathrm{f}(x) = x^3 - x^7 + 1$, which has a root close to 1.1.



(a) Calculate $\mathrm{f}'(0.8)$.

(b) With reference to this value, explain why the Newton-Raphson iteration, with $x_0 = 0.8$, takes a long time to converge to the root of $x^3 - x^7 + 1 = 0$.

1619. The letters of the words HEALTH are jumbled up and rearranged at random.

(a) Explain why there are 5! arrangements where the pairing EA appears in that order.

(b) Hence, find the number of arrangements where the vowels are next to each other.

1620. Sketch the following graphs, for $k \in \mathbb{N}$:

(a) $y = x^{\frac{1}{2k}}$,

(b) $y = x^{\frac{1}{2k+1}}$.

1621. Two (x, y) unit circles are drawn: $x^2 + y^2 = 1$ and $x^2 + (y-4)^2 = 1$. A third unit circle is then drawn so that all three circles are equidistant. Find the two possible equations of the third circle.

1622. By rewriting 2 as e^k, find the factor by which areas are scaled when the graph $y = e^x$ is transformed to $y = 2^x$.

1623. It is given that the expression $x^2 y$ is constant. Find $\frac{dy}{dx}$ in simplified terms of x and y.

1624. Take $g = 10$ in this question.

A particle is projected vertically upwards from ground level at speed 20 ms^{-1}. Two seconds later, another particle is projected in the same manner. Find the height at which the two particles collide.

1625. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning real numbers x and y:

① $y = \log_2 x$,

② $x = 2^y$.

1626. An equilateral triangle is being enlarged. Its area is $\sqrt{48}$ cm^2, and is increasing at a rate of $\sqrt{3}$ cm^2/s.

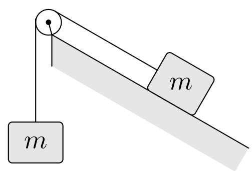
(a) Write the perimeter in terms of the area.

(b) Hence, show that, at this particular instant, the perimeter is increasing at $\frac{3}{2}$ cm/s.

1627. State, with a reason, whether the following gives a well-defined function:

$$\mathrm{g} : \begin{cases} [-1, 1] \mapsto \mathbb{R} \\ x \mapsto \dfrac{1}{\sqrt{1 - x^2}}. \end{cases}$$

1628. Two masses are connected by a light, inextensible string, which is passed over a smooth, light, fixed pulley as shown in the diagram. One mass sits on a rough slope of inclination 30°. The system is in equilibrium.



Find the set of possible values of μ.

1629. In this question, do not use a calculator.

If $y = \tan^2 3x$, determine the value of $\left.\dfrac{dy}{dx}\right|_{x = \frac{1}{18}\pi}$

1630. The IQ scores of a large population are modelled with a normal distribution $X \sim N(100, 15^2)$. A random sample of size 40 is taken.

(a) Write down the distribution of $\bar{X}$, the mean score of the sample.

(b) Find $\mathbb{P}(|\bar{X} - 100| > 5)$.

1631. Determine the number of roots of the equation

$$(x^2 + 4x + 4)(x^4 + 4x^2 + 4) = 0.$$

1632. At take-off, a fighter jet accelerates at $(\mathbf{i} + 2\mathbf{j} + \mathbf{k})g$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors with $\mathbf{k}$ vertical. Find the magnitude of the contact force on a pilot of mass m.

1633. Find the range of $x \mapsto (e^x - 1)^3$..

1634. The interior angles of an irregular hexagon are in geometric progression, and the largest is $\frac{128\pi}{63}$ rad.

(a) Considering the largest angle as the first term a, show that the common ratio r satisfies

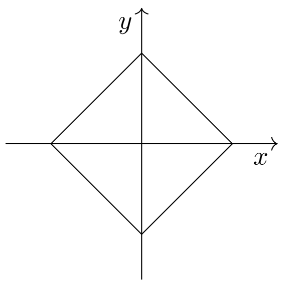
$$128r^6 - 252r + 124 = 0.$$

(b) Using Newton-Raphson, solve this equation.

(c) Hence, show that the smallest angle is $\frac{4}{63}\pi$ rad.

1635. Show that, if $4x^5 - 7x^3 + 2x + 1$ is expressed as $(2x-1)\,\mathrm{f}(x)$, then $\mathrm{f}(x)$ is not polynomial.

1636. The diagram shows the graph $|x| + |y| = 1$:



This graph is translated by vector $a\mathbf{i} + b\mathbf{j}$. Write down the equation of the new graph.

1637. Show that $\displaystyle\int_0^1 \frac{35(1+x)^3}{\sqrt{x}}\,dx = 192.$

1638. Simplify $\big(-2, 2\big] \cap \big[-3 - k^2, 3 + k^2\big)$, for $k \in \mathbb{R}$.

1639. A binomial hypothesis test has acceptance region $\{3, 4, ..., 12, 13\}$. A student writes: "In the sample, $x = 2$, which lies in the critical region, so there is sufficient evidence to reject the null hypothesis." State, with a reason, whether this sentence is valid.

1640. Velocity v takes the following values at times t:

t	0	2	4	6
v	0	1.2	4.8	v_6

(a) Show that the data above is inconsistent with an assumption of constant acceleration.

(b) Show that the data is consistent with $a \propto t$, and find v_6 according to this model.

1641. Find all pairs of values (x, y) which satisfy

$$2\sqrt{x} + \sqrt{y} = 15,$$
$$4\sqrt{y} - \sqrt{x} = 24.$$

1642. For some real constant k, an equation E is given as $kx^2 + (k+1)x + (k+2) = 0$. Determine **three** values of k for which E has exactly one real root.

1643. "The line $y = x$ is normal to the curve $y = 4x - x^2$." True or false?

1644. For a game, a circle is to be divided up into three sectors, such that the probabilities of a needle, spun at the centre of the circle, landing on the three sectors are in the ratio $1 : 3 : 5$.

(a) Find the angles subtended by the sectors.

(b) Show that the probability of two successive spins giving different results is $\frac{46}{81}$.

1645. Factorise $3e^{4x} - 1 - 2e^{2x}$.

1646. A number theorist sets up a function to answer the following question: "If a real number x can be expressed as p/q where $p, q \in \mathbb{Z}$ have no common factors, what are p and q?"

Explain why the largest possible domain for the function is $\mathbb{Q} \setminus \{0\}$.

1647. Prove that, if a projectile is launched from and returns to ground level, then the angle below the horizontal at which it lands is equal to the angle above the horizontal with which it was launched.

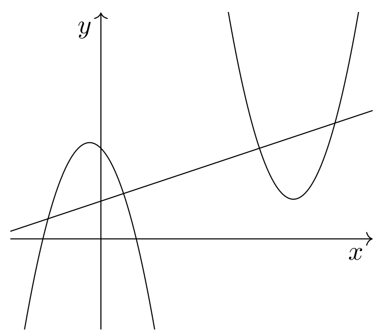
1648. True or false?

(a) $x^2 + 1$ has no linear factors,

(b) $x^2 + x + 1$ has no linear factors,

(c) $x^2 + 2x + 1$ has no linear factors.

1649. The shortest path between two smooth curves is always normal to both. This question concerns the curves $y = 4 - x - x^2$ and $y = x^2 - 17x + 74$.



(a) Show that $3y = x + 5$ is normal to both curves.

(b) Hence, show that the length of the shortest path between the curves is $\sqrt{40}$.

1650. The following problem has been set in a textbook: "Solve the simultaneous equations $4x + 6y = 1$ and $y = -\frac{2}{3}x + \frac{1}{6}$." A mathematics student has been unable to find a unique answer. Explain what the correct interpretation of this fact is, and give the correct solution set.

1651. Three circles C_i have equations

$$\begin{aligned} C_1 &: (x-1)^2 + (y+2)^2 = 1 \\ C_2 &: x^2 + y^2 = 2, \\ C_3 &: (x+2)^2 + (y+2)^2 = 10. \end{aligned}$$

Show that the circles are concurrent.

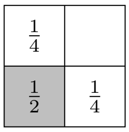
1652. A function is given, for $x \in \mathbb{R}$, by

$$\mathrm{f} : x \mapsto \ln\big(1 + x^2\big).$$

(a) Find $\mathrm{f}(1)$ and $\mathrm{f}'(1)$.

(b) Use these values to show that, at $x = 1$, the linear function that best approximates $\mathrm{f}(x)$ is $\mathrm{g}(x) = x - 1 + \ln 2$.

1653. A computer programmer is modelling a random walk on a 2×2 grid. At any iteration, one square of the four is shaded. At the next iteration, the shading either stays where it is, or moves to an adjacent square. The probabilities, relative to the current position, are as shown in the diagram.



Find the probability that, after two iterations,

(a) the shading is at the opposite corner,

(b) the shading is where it began.

1654. By simplifying $a^{\log_a xy}$ and $a^{\log_a x + \log_a y}$, prove the law of logarithms

$$\log_a xy \equiv \log_a x + \log_a y.$$

1655. A sample $\{x_i\}$ of size n has mean $\bar{x}$ and variance s^2. In terms of these quantities, find the mean of

(a) $\{2x_i + 3\}$,

(b) $\{x_i^2\}$.

1656. For aid distribution, bales of supplies, each of mass m kg, are being dropped from an aeroplane. The aeroplane is in level flight travelling at a constant 40 ms^{-1}. Assume that, once the bales are in the air, air resistance due to their horizontal speed will generate a constant horizontal force of $2m$ N on each one. The bales spend 8 seconds in the air.

(a) Find the horizontal distance between dropping point and landing point.

(b) The bales will split apart if they land with a speed greater than 30 ms^{-1}. What is the greatest vertical speed with which they can safely land?

1657. You are given that the two lines $3x + ky = 0$ and $2x = (k^2 - 1)y$ are perpendicular to one another. Determine all possible values of k.

1658. Find the value of

(a) $\displaystyle\lim_{x\to\infty} \frac{e^x}{e^x + e^{-x}}$,

(b) $\displaystyle\lim_{x\to-\infty} \frac{e^x}{e^x + e^{-x}}$.

1659. Solve $49^x - 56 \cdot 7^x + 343 = 0$.

1660. For real numbers $a, b > 0$, the quantities $\frac{1}{2}(a+b)$ and $\sqrt{ab}$ are known as the *arithmetic mean* and *geometric mean*. Explain the relationship of these means to arithmetic and geometric progressions.

1661. Three variables are related as follows:

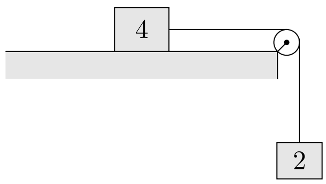
$$\begin{aligned} b^2x - y &= 0, \\ x + by &= 1. \end{aligned}$$

Show that $y = x^{\frac{1}{3}}(x-1)^{\frac{2}{3}}$.

1662. Two dice have been rolled, with scores X_1 and X_2. Given that $|X_1 - X_2| = 1$, find $\mathbb{P}(X_1 + X_2) = 7$.

1663. You are given that $\frac{d}{dx}(x + y^2 - 3) = x + 1$. Find $\frac{dy}{dx}$ in terms of x and y.

1664. A smooth pulley system of two blocks is set up on a table as depicted below. Masses are given in kg. The hanging block accelerates at a.



(a) State two assumptions beyond smoothness which are necessary to find the acceleration of the system with the information given.

(b) Making these assumptions, find a.

(c) Find the force the pulley exerts on the string.

1665. Show that the range of the function $x \mapsto \sin x$ is the same over any domain of the form $[k, k + 2\pi]$.

1666. One of the following statements is true; the other is not. Prove the one and disprove the other.

(a) $x^3 = x \implies x = 0$,

(b) $x^3 = -x \implies x = 0$.

1667. Sketch the graph $y = \dfrac{|x|}{x}$.

1668. Prove that the quadratic (x^2+4) is not a factor of $x^4+x^3+5x^2+4x-2$.

1669. A function f is defined over $\mathbb{R}$ by

$$\mathrm{f}(x) = \frac{1}{8x^4+24x^2+28}.$$

(a) Show that the range of the denominator of the fraction is $[28,\infty)$.

(b) Hence, find the range of f.

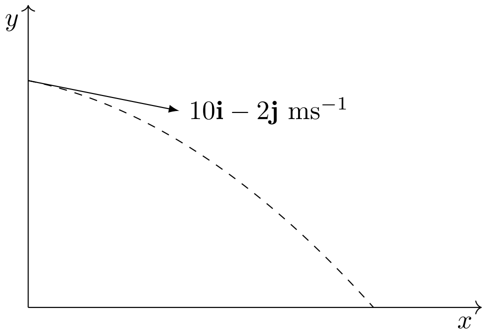
1670. Four cards are pulled from a standard deck of 52. Find the probability that

(a) all four cards are red.

(b) two cards are red and two are black.

1671. Two functions f and g are such that $\mathrm{f}'(x) - \mathrm{g}'(x)$ is linear in x. Prove that the equation $\mathrm{f}(x) = \mathrm{g}(x)$ has at most two roots.

1672. The usual projectile model assumes negligible air resistance. A particle is projected at $10\mathbf{i}-2\mathbf{j}$ ms^{-1} from 5 metres above flat ground, where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical unit vectors.



(a) Determine the horizontal range, if

i. air resistance is neglected as usual,

ii. constant horizontal and vertical resistances of $\frac{1}{5}mg$ are instead assumed.

(b) Comment on these values.

1673. The monic quadratic function f is invertible over either of the domains $(-\infty,2]$ or $[2,\infty)$ with the codomain $[3,\infty)$.

(a) Give the coordinates of the vertex of $y=\mathrm{f}(x)$.

(b) Hence, find $\mathrm{f}(0)$.

1674. (a) Show that $x^7-x^2=0$ has one single root, one double root, and no other real roots.

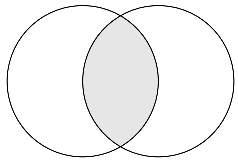
(b) Hence, sketch $y=x^7-x^2$.

(c) Hence, solve the inequality $x^7-x^2 \geq 0$, giving your answer in set notation.

1675. Prove that a polynomial of order n can have at most $n-2$ points of inflection.

1676. A graph, whose equation is $\mathrm{f}(x)=\mathrm{g}(y)$ for some functions f and g, is reflected in $y=x$, then in $x=0$, then in $y=0$. Find the equation of the transformed graph.

1677. In the figure below, the circles, both of which have radius r, pass through each other's centres.



Show that the shaded intersection has area

$$A_{\text{int}} = r^2\left(\tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}\right).$$

1678. A quintic function is defined as $\mathrm{f}(x)=x^5-x$. At each of its roots, find, as a function of x, the best linear approximation to $\mathrm{f}(x)=x^5-x$.

1679. Prove rigorously that $\displaystyle\lim_{x\to\infty}\frac{x+1}{2x^2+1}=0$.

1680. A parabola $y=ax^2+bx+c$ is increasing on $(-\infty,0)$, decreasing on $(0,\infty)$, and has distinct roots. Give conditions on the constants a,b,c and sketch the curve.

1681. Solve for a in $(2a+1)^4+(2a-1)^4=82$.

1682. State that the following holds, or explain why not: "Knowing that an object is in equilibrium is both necessary and sufficient for the resultant moment acting on it to be zero."

1683. Show that the curves $4y=x^2+1$ and $4x=y^2+1$ are not tangent.

1684. A quadratic sequence has three consecutive terms $12, 16, 22$. Find the values of the two consecutive terms of the sequence which differ by sixty.

1685. Make x the subject of $y=\dfrac{x^3-1}{x^3+1}$.

1686. True or false?

(a) $x\in A \implies x\in A\cup B$,

(b) $x\notin A \implies x\notin A\cap B$,

(c) $x\notin A \implies x\notin A\cup B$.

1687. Write the following in simplified interval notation:

$$\{x\in\mathbb{R} : |x-1|\leq 1\} \cup \{x\in\mathbb{R} : |x+1|<\tfrac{3}{2}\}.$$

1688. Find the values of the constants A, B, C, D such that the following identity holds:

$$\frac{8x^3 - 6x^2 - x + A}{2x - 1} \equiv Bx^2 + Cx + D.$$

1689. Show that the following curve has no asymptotes:

$$y = \frac{1}{\cos^2 x + \cos x + 1}.$$

1690. By finding equations of perpendicular bisectors, or otherwise, show that a quadrilateral with vertices at $(8, 2)$, $(6, 4)$, $(12, 10)$ and $(14, 4)$ is cyclic.

1691. A square of side length 1 lies in a horizontal (x, y) plane. Three forces act, as in the following table:

(x, y)	Force
$(1, 1)$	$-2\mathbf{i}$ N
$(1, 0)$	$3\mathbf{j}$ N
$(0, 1/2)$	$\mathbf{F}$ N.

The resultant moment of these three forces is zero.

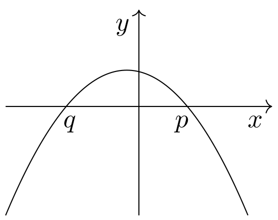
(a) Draw a force diagram.

(b) Show that equilibrium cannot be maintained.

1692. Find $\displaystyle\int \cos\left(4x + \frac{\pi}{2}\right)\, dx$.

1693. The two lines $2x + 3y = 5$ and $ax + by = c$ do not intersect. Determine the value of $2bc - 3ac$.

1694. A quadratic graph $y = ax^2 + bx + c$ is shown below. It is given that $q < 0 < p$, and $|p| < |q|$.



State, with a reason in (b), whether the following facts are necessarily true:

(a) "a is negative",

(b) "b is negative",

(c) "c is positive".

1695. A hand of five cards is dealt from a standard deck, with ace counting high. Let p be the probability that the hand is a straight flush, i.e. a set of five consecutive cards all of the same suit. Show that

$$p = \frac{36 \times 5! \times 47!}{52!}.$$

1696. Show that the graph $y = 10^x$ is transformed into that of $y = e^{x+2}$ by a stretch in the x direction and a stretch in the y direction. Give the exact value of each scale factor.

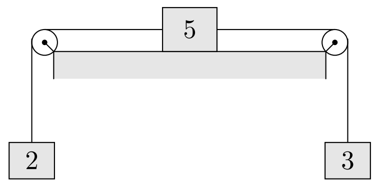
1697. State, with a reason, whether the following hold:

(a) $\displaystyle\sum_{r=1}^{n} ku_r \equiv k\sum_{r=1}^{n} u_r$,

(b) $\displaystyle\sum_{k=1}^{n} ku_k \equiv k\sum_{k=1}^{n} u_k$,

(c) $\displaystyle\sum_{r=1}^{n} nu_r \equiv n\sum_{r=1}^{n} u_r$.

1698. A smooth pulley system of three masses, given in kg, is set up on a table as depicted below.



(a) Explain the assumptions necessary to model

i. the accelerations as equal in magnitude,

ii. there being exactly two values for tension.

(b) Making all necessary assumptions, draw force diagrams for the masses.

(c) Hence, find the acceleration of the system.

1699. State, with a reason, whether the following curves intersect the line $x = \frac{\pi}{2}$:

(a) $y = \operatorname{cosec} x$,

(b) $y = \sec x$,

(c) $y = \cot x$.

1700. Two vertices of an equilateral triangle lie at $(0, 0)$ and $(2, 1)$. Show that the other vertex lies on the line $2y + 4x = 5$.

—— End of 17th Hundred ——